Optimal Control of Parallel Make-To-Stock Queues
in an Assembly System

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Abstract

This paper considers an assembly system in which components are produced to stock but the final assembly of the product is made to order. Different components are produced on separate dedicated facilities. The production process for each component is modelled as an exponential make-to-stock queue and the assembly process is assumed instantaneous. Demand arrivals are modelled by a Poisson process. At each demand arrival, if all components have inventory in stock, a unit of the product is assembled to satisfy the demand; otherwise the demand is backordered which accrues penalty cost until it is satisfied. On the other hand components in inventory carry holding costs. The problem is to control the component production, based on the inventory levels of all the components in the system, to minimize the discounted total cost. We prove that the optimal control policy is dynamic in that each component should be produced up to an inventory level dependent on the inventory levels of the other components. Furthermore we show that this level is upper bounded so that production of each component is halted when its inventory reaches an absolutely high level independent of the inventory levels of the other components. We also report numerical examples and in particular observe that even the best static base stock policy can perform infinitely worse than the optimal dynamic policy.

(Assembly System, Make-to-Stock Queues, Make-to-Order Assembly, Dynamic Optimal Control, Base Stock Policy)

1 Introduction

Consider an assembly system that produces multiple components and assembles them into a final product. The components are produced to stock on separate dedicated facilities. The
component production processes are random due to uncertainties in production time and/or yield. When all components have inventory in stock it is possible to assemble a unit of finished product by taking one unit from each component inventory. The assembly process is assumed 100% reliable and requiring negligible amount of time.

Demand for the finished product arrives in single units. At arrival if all components have inventory in stock the assembly process is executed and a unit of finished product is output to satisfy the demand. Otherwise if some components have zero inventory available in stock the demand is backordered until those components are produced. A profit is realized For each demand satisfied, a penalty cost is levied for demands backordered per unit time. On the other hand the components in inventory carry inventory holding costs per unit time. The problem we study in the paper is to optimally control the component production processes so that the discounted total cost of the assembly system is minimized.

The system is a model for some assemble-to-order systems, on which a comprehensive survey of the research literature is done by Song and Zipkin (2003) (see also Cheng, et al. 2002, Lu, et al. 2005, Song and Yao 2002 and Xu 2001). As born out most of the existing works focuses on performance evaluation of some predefined control policies with the assemble-to-order systems. In particular, Song et al. (1999) investigated performances of base stock polices under various order fulfilling schemes for systems more general than what we study in the paper. We are making an attempt at the optimal control problem by considering a relatively simple type of system.

The optimal control policy we determine in our work turns out to be a type of dynamic “curve” control. Namely, each component should be produced up to an inventory level which is dependent on the inventory levels of the other components. Roughly, the higher the other inventory levels of the other components, the higher the produce-up-to level for the concerned component. Furthermore we show that the control “curve” is upper bounded for each component. Thus for each component there is an inventory level over which production of the component is halted no matter what the inventory levels are of the other components. Also we verify the intuition that, if one component has zero inventory while there are backordered demands and all the other components have positive inventory, the zero inventory component must be produced.

Although the dynamic optimal control policy of “curve” type is not surprising to many researchers, we believe our contributions are two folds. Firstly, we give a formal proof and characterization of the control curve and obtain some interesting asymptotic properties of the policy. Secondly, we learn from an extensive numerical study that the best base stock policy can perform infinitely worse than the optimal dynamic policy. Therefore we provide a cautionary note to the research literature’s hitherto attention focus on the base stock policies.
Such policies may be easy to manage in practice and easy to analyze in theory, but could lead to significant under performance for the system.

The remainder of the paper is organized as follows. Section 2 reviews the literature on the optimal control of make-to-stock queues. Section 3 describes the assembly system and presents the precise formulation of the optimal control problem that we study in the paper. Section 4 characterizes the optimal control policy and proves several asymptotic properties. Section 5 reports results from the numerical study. Section 6 generalizes the optimal control policy to multi-component system. Finally, Section 7 concludes the paper with a summary of results and a discussion on future research.

2 Literature Review

Make-to-stock queues were first employed to model production systems by Gavish and Graves (1980) and Soble (1982), who modelled single product single facility production systems by $M/D/1$ queues and derived base stock optimal control policies. For $M/M/1$ make-to-stock queues the optimality of the base stock policy was first proved in Li (1988) for the discounted total cost. The result was extended by Feng and Yan (2000) to allow random break downs of the production facility. Later Feng and Xiao (2002) proved the same optimality result under long run average cost.

However most of the literature on the optimal control of make-to-stock queues have been concerned with multiple product single facility systems. For these systems the control problem involves an additional issue of product sequencing when the production is on as well as the issue of turning production on and off. Zheng and Zipkin (1990) showed that in the case of two symmetric products the sequencing policy that always produce the product with the longest backorder queue is better than the policy of first-come-first-serve. Wein (1992) investigated the optimal control problem of a general multiclass make-to-stock queue by following the Brownian approximation approach, and proposed a hedging point policy which is a generalization of the base stock policy. Ha (1997a) analytically proved some optimality properties of the hedging point policy. In particular he showed that, for a make-to-stock queue with two products facing Poisson demand processes and requiring identically distributed exponential production times, a dynamic version of the hedging policy is optimal under the objective of minimizing the discounted total cost. The dynamic hedging point policy is of “curve” control type: one curve determines the on-off region for production and the other curve determines when to switch the product to be produced. Further characterization of the optimal switching curve was obtained by De Vericourt et al. (2000). Our work follows very much the same line of analysis as in Ha (1997a) and establishes similar structure properties of the optimal control.
There are also numerical studies on heuristics, e.g., Veatch and Wein (1996) and Peña-Perez and Zipkin (1997), and asymptotic studies on various limiting schemes, e.g., Glasserman and Wang (1998) and Bertsimas and Paschalidis (2001). Other types of control problems for multiclass make-to-stock queues have also been studied. For example, Ha (1997b, 1997c) considered the stock rationing issue, Carr and Duenyas (2000) included the issue of admission control, and Caldenty and Wein (2003) added a new dimension of revenue management.

A few works are closely related to ours. One by Chen et al. (1993) concerns the same single product assembly system but in the setting that the assembly is done as soon as there are units in inventory for all the components and all assembled products yield profit. In this case the optimal controls on the component production processes are threshold type with the threshold for each component independent of the others. In Hsu et al. (2005), they also analyze the optimal stocking quantities for components of an assemble-to-order product. In their work, the manufacture faces uncertain and one time demand (and thus no queueing of demand), and need to make pricing for the final product.

3 System Description and Problem Formulation

The assembly system we study is illustrated in Figure 1 below for a product with two components. In fact we will provide detailed proofs for the properties of the dynamic optimal control policy only for the case of two components, and will highlight their extensions to multiple components.

In the system of Figure 1 the components are produced by two separate facilities. The production processes take exponential amount of time with a rate of $\mu_i$ for component $i$, $i = 1, 2$. The finished product is assembled from one unit of each component. The assembly process is 100% reliable and takes 0 amount of time. Demand for the finished product arrives according to a Poisson process with rate $\lambda$. We assume that any unmet demand on arrival is
backordered, i.e., there is no rejection of demand. Therefore, we assume \( \mu_i > \lambda, \ i = 1, 2 \), to make sure the system’s long term stability.

At a demand arrival, if both components have inventories in stock, the assembly process is executed to output one unit of finished product to satisfy the demand. Otherwise, the demand is added to a waiting queue which is served in first-come-first-serve order. The half-finished components in inventory carry a holding cost of \( c_i^+ \) per unit component per unit time, \( i = 1, 2 \). The backordered demands carry a penalty cost of \( c^- \) per waiting demand per unit time. The optimal control problem for the system is to manage the production processes of the two components, based on the component inventories and demand backordering, so that the discounted total cost over infinite time horizon is minimized.

We specify a control policy for the system by a production rate function \( \mu(t) = (\mu_1(t), \mu_2(t)) \) for all time \( t > 0 \), where \( \mu_i(t) = 0 \) (production off) or \( \mu_i \) (production on). The policy is said non-anticipatory if, at any \( t > 0 \), \( \mu(t) \) depends only on information prior to \( t \). Let \( \mathcal{U} \) be the collection of all such non-anticipatory control policies. We use the components’ inventory positions as the state variables for the system, where a component’s inventory position is its inventory in stock minus the number of backordered demands. Given a control policy \( u = \{\mu(t) : t > 0\} \in \mathcal{U} \), we denote the components inventory positions at time \( t > 0 \) by \( x^u(t) = (x_1^u(t), x_2^u(t)) \). Then starting from an initial state \( x = (x_1, x_2) \) the discounted total cost over infinite time horizon is given by

\[
J^u(x) = E \left[ \int_0^\infty e^{-\gamma t} c(x(t)) dt \right].
\]

Here, \( \gamma \) is the discount factor, and \( c(x(t)) \) is the cost function and given as

\[
c(x(t)) = \sum_{i=1}^{2} c_i^+ w_i(t) + c^- b(t),
\]

where \( b(t) \) is the amount of backordered demands at time \( t \), i.e.,

\[
b(t) = -\min\{x_1(t), x_2(t), 0\};
\]

and \( w_i(t), i = 1, 2 \), is the on hands inventory of component \( i \) at time \( t \), i.e.,

\[
w_i(t) = x_i(t) + b(t).
\]

It can be confirmed that the cost \( c(x_1, x_2) \) is a convex and submodular function.

A policy \( u^* \in \mathcal{U} \) is said optimal if it minimizes \( J^u(x) \) for any starting state \( x \), i.e., it solves the following optimization problem

\[
J^*(x) = \inf_{u \in \mathcal{U}} J^u(x), \text{ for } x \in \mathbb{Z}^2.
\]
Without loss of generality, we can uniformize the instantaneous rates of changes for the Markov decision process by assuming $\gamma + \lambda + \mu_1 + \mu_2 = 1$. Then, we obtain the Hamilton-Jacobi condition for this Markov decision process optimal control problem as follows (cf. Chapter 7 of Ross 1970),

$$J(x_1, x_2) = c(x_1, x_2) + \lambda J(x_1 - 1, x_2 - 1) + (\mu_1 + \mu_2)J(x_1, x_2) + \frac{2}{i=1} \min[0, \mu_i D_i J(x_1, x_2)], \quad (2)$$

where, for any function $v(x_1, x_2)$, $D_1 v(x_1, x_2) = v(x_1 + 1, x_2) - v(x_1, x_2)$, and $D_2 v(x_1, x_2) = v(x_1, x_2 + 1) - v(x_1, x_2)$.

4 Dynamic Optimal Control and Its Properties

In this section we characterize the solution to the Hamilton-Jacobi condition (2) and present some of its asymptotic properties. All proofs are collected in the appendix. For notational convenience, we define an operator $T$ on the set of real-valued functions $v(x_1, x_2)$ as follows,

$$Tv(x_1, x_2) = c(x_1, x_2) + \lambda v(x_1 - 1, x_2 - 1) + (\mu_1 + \mu_2)v(x_1, x_2) + \frac{2}{i=1} \min[0, \mu_i D_i v(x_1, x_2)]. \quad (3)$$

Let $V$ be the set of functions defined on $Z^2$ such that if $v \in V$, then:

(a) **Submodularity** ($D_{12} v \leq 0$): $D_1 v(x_1, x_2)$ is nonincreasing in $x_2$ and $D_2 v(x_1, x_2)$ is nonincreasing in $x_1$;

(b) **Convexity** ($D_{11} v \geq 0, D_{22} v \geq 0$): $D_1 v(x_1, x_2)$ is nondecreasing in $x_1$ and $D_2 v(x_1, x_2)$ is nondecreasing in $x_2$.

The following lemma shows that submodularity and convexity are preserved under the operator $T$:

**Lemma 1** If $v \in V$, then,

a) $\min[v(x_1 + 1, x_2), v(x_1, x_2)] \in V$,

b) $\min[v(x_1, x_2 + 1), v(x_1, x_2)] \in V$, and

c) $Tv(x_1, x_2) \in V$.

Lemma 1 enables us to characterize the solution to the Hamilton-Jacobi condition (2) as in the following theorem:

**Theorem 1** a) The optimal cost function $J(x)$ is submodular and convex.
b) Let

\[ B_1(x_2) = \min\{x_1 : D_1J(x_1, x_2) \geq 0\}, \text{ and} \]
\[ B_2(x_1) = \min\{x_2 : D_2J(x_1, x_2) \geq 0\}. \]

Then, at state \( x = (x_1, x_2) \), it is optimal to stop production of component 1 if \( x_1 \geq B_1(x_2) \), and to stop production of component 2 if \( x_2 \geq B_2(x_1) \).

c) \( B_1(x_2) \) is increasing in \( x_2 \), and \( B_2(x_1) \) is increasing in \( x_1 \).

Theorem 1 reveals that the optimal control policy is dynamic and “curve” type. \( B_1(x_2) \) is the produce-up-to level curve for component 1, and it depends on the inventory level of component 2, and vice versa. We state in the following some asymptotic properties of the control curves \( B_1(x_2) \) and \( B_2(x_1) \).

Firstly, we verify the intuition that, if one component’s inventory is zero while there are backordered demands and the other component has higher inventory position, then the former component must be produced. Formally this is expressed as in the following theorem:

**Theorem 2** \( B_1(x_2) > \min(0, x_2) \) and \( B_2(x_1) > \min(0, x_1) \).

Next, based on submodularity and convexity, we can conclude that both \( D_1J(x_1, x_2) \) and \( D_2J(x_1, x_2) \) are monotone over \( x_1 \) and \( x_2 \), and therefore, the following limits exist:

\[
\lim_{x_i \to \infty} D_iJ(x_1, x_2) = f_i(x_j), \quad i, j = 1, 2, i \neq j;
\]
\[
\lim_{x_i \to \infty} D_jJ(x_1, x_2) = g_i(x_j), \quad i, j = 1, 2, i \neq j.
\]

We further show that this convergence is uniform. A function \( v(x_1, x_2) \) converges to \( h(x_2) \) uniformly in \( x_2 \) as \( x_1 \to \infty \), if for any given \( \epsilon > 0 \) there exists a constant \( X_1 \), independent of \( X_2 \), such that for all \( x_1 > X_1 \), \( |v(x_1, x_2) - h(x_2)| < \epsilon \).

**Theorem 3**

\[
D_iJ(x_1, x_2) \to f_i(x_j) = \frac{c_1^+}{\gamma} \quad \text{uniformly in } x_j \text{ as } x_i \to \infty, \text{ for } i, j = 1, 2, i \neq j;
\]
\[
D_jJ(x_1, x_2) \to g_i(x_j) < \infty \quad \text{uniformly in } x_j \text{ as } x_i \to \infty, \text{ for } i, j = 1, 2, i \neq j.
\]

Theorem 3 indicates that the value function under the optimal policy is asymptotically linear. In particular, the intuition for the first limit is that, suppose component 1’s inventory is very high, having one more unit of component 1 would incur an additional cost \( c_1^+ \) per unit time. In the limit, the net present value of this additional cost is \( c_1^+ \int_0^\infty e^{-\gamma t} dt = \frac{c_1^+}{\gamma} \).

More importantly, Theorem 3 leads to the boundedness of the optimal control curves as stated in the following corollary:
Corollary 1 There exists a constant \( \bar{B}_1 \) (independent of \( x_2 \)) such that \( B_1(x_2) \leq \bar{B}_1 \) for all \( x_2 \); and there exists a constant \( \bar{B}_2 \) (independent of \( x_1 \)) such that \( B_2(x_1) \leq \bar{B}_2 \) for all \( x_1 \).

Finally, we have that for symmetric components the two optimal control curves are the same:

Theorem 4 If \( c_1^+ = c_2^+ \) and \( \mu_1 = \mu_2 \), then \( B_1(\cdot) = B_2(\cdot) \).

However below we will show by a numerical example that, even for symmetric components, it is not necessarily true that \( B_1(\cdot) = B_2(\cdot) = \text{const} \). Thus, a static base stock policy cannot be optimal, and actually we will show that even the best static base stock policy can perform infinitely worse than the optimal dynamic policy.

5 Numerical Examples

In this section, we compute the optimal dynamic control policy for two numerical examples, one with symmetric components and one with asymmetric components. For each example we also find through brute force search the best static base stock policy. We remark that the dynamic and static policies differ mostly at the instants when both components are backordered but one is more than the other. The static policy will try to bring both component’s inventory up to the predefined base stock levels that are independent of the inventory state of the other component. However, the dynamic policy will first hold production of the less backordered component since the more of it will not reduce the product backordering queue but only increase inventory. As a result, the dynamic policy in the long run incurs less component inventory holding costs.

We first evaluate a simple example of a single product with two symmetric components. The product demand arrival rate is \( \lambda = 0.29 \). The two components have the same production rate \( \mu_1 = \mu_2 = 0.35 \) and the same inventory holding cost \( c_1^+ = c_2^+ = 1 \). The backorder cost is \( c^- = -1 \) and the discount rate \( \gamma = 0.01 \). The optimal dynamic control curves are shown in Figure 2. The best static base stock policy for the example is to set base stock levels at \( S_1 = S_2 = 2 \), under which the system cost is higher than the optimal dynamic base stock policy by 9.52%.

Not surprisingly performance difference between the two policies increases with increasing load factor of the system as indicated in Figure 3. Here, we use the same cost parameters as above but increase the system load factor \( \rho = \lambda / \mu_1 = \lambda / \mu_2 \). In all instances, we uniformize the rate parameters so that the uniformization equation, i.e., \( \lambda + \gamma + \mu_1 + \mu_2 = 1 \), holds.

We also expect that the performance gap between the two policies diminish with increasing backorder cost \( c^- \) as indicated in Figure 4, in which all the parameters are the same as in Figure...
Figure 2: The Dynamic Optimal Control for the Symmetric System

Figure 3: Percentage Performance Difference Between the Optimal Dynamic Control and the Best Base Stock Policy at different load levels
1 except that $c^-$ is varied. This is because for large $c^-$ the base stock levels set by both the static and dynamic are so high that the chances of having backordering queue are extremely small, therefore the two policies will behave quite similarly.

Next, we examine a system with two asymmetric components to illustrate that performance difference between the optimal dynamic policy and the static base stock can be arbitrarily large. The two components’ production rates are $\mu_1 = 0.0105$, and $\mu_2 = 0.979$, and inventory holding costs are $c_1^+ = 10$ and $c_2^+ = 100$. The product demand rate is $\lambda = 0.01$ with backorder cost $c^- = 100$. The discount rate is $\gamma = 0.0005$. The optimal dynamic control curves are shown in Figure 5. The best static base stock policy for the example is to set base stock level for component 1 at $S_1 = 11$ and for component 2 at $S_2 = 0$, under which the system cost is higher than the optimal dynamic base stock policy by 88.2%.

We observe that in this system component 2 has a faster production rate, so when both components are backordered the static policy might build up a large queue for component 2 when component 1 is being produced slowly. Thus we experimented with increasing $c_2^+$ and increasing $\mu_2/\mu_1$ while keeping all other parameters the same as in Figure 5. The results are shown in Figure 6, which indicates that indeed performance difference becomes ever larger with larger $c_2^+$. It also confirms the expectation that the difference gap is wider with higher $\mu_2/\mu_1$ as in this case faster over producing of less backordered component leads to higher inventory holding costs.
Figure 5: The Dynamic Optimal Control for the Asymmetric System

Figure 6: Percentage Performance Difference Between the Optimal Dynamic Control and the Best Base Stock Policy for different $\frac{\mu_2}{\mu_1}$ and component 2’s inventory holding cost $c_2^i$
6 Extensions to Multiple Component Systems

The results of Section 4 can be generalized to multiple component systems due to additivity of the cost terms in the objective function $J^u(x)$ over the $n$ components, where $x = (x_1, \ldots, x_n)$. For convenience, denote $S_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. We state the extensions but omit their proofs.

**Theorem 5**  a) The optimal cost function $J(x)$ is submodular and convex;

b) Let

$$B_i(S_i) = \min\{x_i : D_i J(x) \geq 0\},$$

at state $x$, it is always optimal to stop production of component $i$ if its inventory level is at or above $B_i(S_i)$;

c) $B_i(S_i)$ is increasing in all $x_j, j \neq i$.

**Theorem 6** It is optimal to produce a component that is backlogged and whose inventory position is lower than all the other components.

**Theorem 7** The optimal cost function $J(x)$ has the following limits, for each component $i$:

$$D_i J(x) \to \frac{c^+}{\gamma} \text{ uniformly in } S_i \text{ as } x_i \to \infty;$$

$$D_j J(x) \to g_{ij}(S_i) \text{ uniformly in } S_i \text{ as } x_i \to \infty, \text{ for all } j \neq i.$$

**Corollary 2** The optimal control curve $B_i(S_i)$ is bounded from above for every component $i$.

7 Conclusion

In this paper, the optimal dynamic control policy is characterized for an assembly system in which components are produced to stock but the final assembly of the product is made to order. Not only we prove that the optimal control policy is dynamic in that each component should be produced up to an inventory level dependent on the inventory levels of the other components, but also we show that this level is upper bounded so that production of each component is halted when its inventory reaches an absolutely high level. We also report numerical examples which indicate that comparing with the optimal dynamic control policy even the best static base stock policy can perform infinitely worse.

The results are established for systems of a single product and the product assembled from one unit each of the components. A natural next step is to consider systems of multiple products and the products assembled from multiple component units. For those systems there can arise the issues of full or partial fulfillment of the product demands and of priority allocation...
of components to the product demands. Technically our models are based on assumptions of exponential production times and Poisson demand process which although are standard for optimal control studies, nevertheless are restrictive, might be extended to allow more general production times and demand arrival processes.

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References


Appendix

Proof of Lemma 1  a) Take any $v \in V$ and let $w$ be a function on $\{0,1\} \times Z^2$ defined by

$$w(u, x_1, x_2) = (1 - u)v(x_1, x_2) + uv(x_1 + 1, x_2)$$

$$= \begin{cases} v(x_1, x_2), & \text{if } u = 0, \\ v(x_1 + 1, x_2), & \text{if } u = 1. \end{cases}$$

Since $v \in V$, for any $u$, $w$ is submodular in $(x_1, x_2)$. Moreover,

$$w(u, x_1, x_2 + 1) - w(u, x_1, x_2) = \begin{cases} D_2v(x_1, x_2), & \text{if } u = 0; \\ D_2v(x_1 + 1, x_2), & \text{if } u = 1, \end{cases}$$

and it is decreasing in $u$. Hence $w$ is submodular in $(u, x_2)$ (see Topkis 1978).

Next we show that

$$g(x_1, x_2) = \min[v(x_1 + 1, x_2), v(x_1, x_2)] = \min_{u \in \{0,1\}} w(u, x_1, x_2)$$

is also submodular and convex. To prove submodularity, we assume $u_1, u_2 \in \{0,1\}$ be such that

$$g(x_1 + 1, x_2) = w(u_1, x_1 + 1, x_2),$$

and

$$g(x_1, x_2 + 1) = w(u_2, x_1, x_2 + 1).$$

Then, we have either (i) or (ii) as follows.

i) $u_1 \geq u_2$ which yields

$$g(x_1 + 1, x_2 + 1) + g(x_1, x_2) \leq w(u_1, x_1 + 1, x_2 + 1) + w(u_2, x_1, x_2)$$

$$\leq [w(u_1, x_1 + 1, x_2) + w(u_2, x_1 + 1, x_2 + 1) - w(u_2, x_1 + 1, x_2)]$$

$$+ w(u_2, x_1, x_2)$$

$$\leq w(u_1, x_1 + 1, x_2) - w(u_2, x_1 + 1, x_2) + w(u_2, x_1, x_2)$$

$$+ [w(u_2, x_1 + 1, x_2) + w(u_2, x_1, x_2 + 1) - w(u_2, x_1, x_2)]$$

$$= w(u_1, x_1 + 1, x_2) + w(u_2, x_1, x_2 + 1)$$

$$= g(x_1 + 1, x_2) + g(x_1, x_2 + 1),$$

where the first inequality follows from the definition of $g$, the second from the submodularity of $w$ in $(u, x_2)$, and the third from the submodularity of $w$ in $(x_1, x_2)$; or
ii) \( u_1 < u_2 \), i.e., \( u_1 = 0 \), and \( u_2 = 1 \), which yields
\[
g(x_1 + 1, x_2 + 1) + g(x_1, x_2) \leq w(1, x_1 + 1, x_2 + 1) + w(0, x_1, x_2)
\]
\[
\leq [w(1, x_1 + 1, x_2) + w(0, x_1 + 1, x_2 + 1) - w(0, x_1 + 1, x_2)]
\]
\[
+ w(0, x_1, x_2)
\]
\[
\leq w(1, x_1 + 1, x_2) - w(0, x_1 + 1, x_2) + w(0, x_1, x_2)
\]
\[
+ [w(0, x_1 + 1, x_2) + w(0, x_1, x_2 + 1) - w(0, x_1, x_2)]
\]
\[
= w(1, x_1 + 1, x_2) + w(0, x_1, x_2 + 1)
\]
\[
= g(x_1, x_2) + g(x_1, x_2 + 2).
\]

Hence in both cases we prove submodularity of \( g \).

To prove convexity of \( g \) we need to check the following two inequalities:
\[
g(x_1, x_2 + 2) + g(x_1, x_2) \geq 2g(x_1, x_2 + 1), \text{ and } \tag{4}
g(x_1 + 2, x_2) + g(x_1, x_2) \geq 2g(x_1 + 1, x_2). \tag{5}
\]

To check inequality (4) we assume \( u_1, u_2 \in \{0, 1\} \) be such that
\[
g(x_1, x_2) = w(u_1, x_1, x_2),
\]
\[
g(x_1, x_2 + 2) = w(u_2, x_1, x_2 + 2).
\]

Then we have either (i) or (ii) as follows.

i) \( u_1 \geq u_2 \) which yields
\[
2g(x_1, x_2 + 1) \leq w(u_1, x_1, x_2 + 1) + w(u_2, x_1, x_2 + 1)
\]
\[
\leq [w(u_2, x_1, x_2 + 1) + w(u_1, x_1, x_2) - w(u_2, x_1, x_2)]
\]
\[
+ w(u_2, x_1, x_2 + 1)
\]
\[
\leq [w(u_2, x_1, x_2 + 2) + w(u_2, x_1, x_2)]
\]
\[
+ w(u_1, x_1, x_2) - w(u_2, x_1, x_2)
\]
\[
= w(u_1, x_1, x_2) + w(u_2, x_1, x_2 + 2)
\]
\[
= g(x_1, x_2) + g(x_1, x_2 + 2),
\]

where the first inequality follows from the definition of \( g \), the second from the submodularity of \( w \) in \((u, x_2)\), and the third from the convexity of \( w \) in \( x_2 \); Or

ii) \( u_1 < u_2 \), i.e., \( u_1 = 0 \) and \( u_2 = 1 \), in which case inequality (4) follows straightforwardly. To check inequality (5) we define
\[
\bar{w}(u, x_1, x_2) = \begin{cases} (1-u)v(x_1, x_2) + w(x_1, x_2 + 1) & \\
v(x_1, x_2), & \text{if } u = 0, \\
v(x_1, x_2 + 1), & \text{if } u = 1.
\end{cases}
\]

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Then, by applying the above steps to \( \bar{w} \) instead of \( w \), we can verify the inequality (5).

b) The proof is similar to that of a) as given above and omitted.

c) It is easy to check that \( c(x_1, x_2) \in V \). Since \( V \) is closed under addition and multiplication by positive scalars, we obtain that \( Tv(x_1, x_2) \in V \).

**Proof of Theorem 1**  Lemma 1 asserts that convexity and submodularity are preserved by operator \( T \). Since the cost function is nonnegative and the control set is finite, by Proposition 13 of Chapter 5 (page 218) in Bertsekas (1987) we can conclude that the optimal cost function \( J(x) \) is also submodular and convex.

Consequently, for any given \( x_2 \), there exists a \( B_1(x_2) \) such that, for any \( x_1 \geq B_1(x_2) \), \( D_1J(x_1, x_2) > 0 \); and for any \( x_1 < B_1(x_2) \), \( D_1J(x_1, x_2) \leq 0 \). Furthermore, by convexity and submodularity of \( J(x_1, x_2) \), \( B_1(x_2) \) is increasing in \( x_2 \). Similar conclusions can be obtained regarding \( B_2(x_1) \).

**Proof of Theorem 2**  Let \( V_1 \) be the subset of \( V \) that contains all the functions with \( D_2v(x_1, x_2) < 0 \) for \( x_1 > x_2 \) and \( x_2 < 0 \). We want to show that \( V_1 \) is closed under operator \( T \). Given a function \( v \in V_1 \), we have

\[
D_2Tv(x_1, x_2) = Tv(x_1, x_2 + 1) - Tv(x_1, x_2)
\]

\[
= c(x_1, x_2 + 1) + \lambda v(x_1 - 1, x_2) + (\mu_1 + \mu_2)v(x_1, x_2 + 1)
+ \min\{0, \mu_1 D_1v(x_1, x_2 + 1)\} + \min\{0, \mu_2 D_2v(x_1, x_2 + 1)\}
- c(x_1, x_2) - \lambda v(x_1 - 1, x_2 - 1) - (\mu_1 + \mu_2)v(x_1, x_2)
- \min\{0, \mu_1 D_1v(x_1, x_2)\} - \min\{0, \mu_2 D_2v(x_1, x_2)\}
\]

\[
= D_2c(x_1, x_2) + \lambda D_2v(x_1 - 1, x_2 - 1) + (\mu_1 + \mu_2)D_2v(x_1, x_2)
+ \min\{0, \mu_1 D_1v(x_1, x_2 + 1)\} + \min\{0, \mu_2 D_2v(x_1, x_2 + 1)\}
- \min\{0, \mu_1 D_1v(x_1, x_2)\} - \min\{0, \mu_2 D_2v(x_1, x_2)\}
\]

\[
= D_2c(x_1, x_2) + \lambda D_2v(x_1 - 1, x_2 - 1)
+ \min\{0, \mu_1 D_1v(x_1, x_2 + 1)\} + \min\{0, \mu_2 D_2v(x_1, x_2 + 1)\}
+ \begin{cases} 
\mu_1[D_2v(x_1, x_2) - D_1v(x_1, x_2)], & \text{if } 0 > D_1v(x_1, x_2), \\
\mu_1D_2v(x_1, x_2), & \text{if } 0 \leq D_1v(x_1, x_2).
\end{cases}
\]

Note that when \( x_1 > x_2 \) and \( x_2 < 0 \), having one more unit of \( x_2 \) reduces both the backorder cost of the product and the holding cost of component 1, so \( D_2c(x_1, x_2) < 0 \). Using the identity

\[
D_2v(x_1, x_2) - D_1v(x_1, x_2) = D_2v(x_1 + 1, x_2) - D_1v(x_1, x_2 + 1),
\]

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Let \( v \) denote vectors \((x_1, x_2) \in \mathbb{R}^2 \). We have the following uniform convergence in \( x_2 \):

\[
D_Tv(x_1, x_2) = D_2c(x_1, x_2) + \lambda D_2v(x_1 - 1, x_2 - 1) + \min[0, \mu_1 D_1v(x_1, x_2 + 1)] + \min[0, \mu_2 D_2v(x_1, x_2 + 1)]
\]

\[
+ \begin{cases} 
\mu_1[D_2v(x_1 + 1, x_2) - D_1v(x_1, x_2 + 1)], & \text{if } 0 > D_1v(x_1, x_2), \\
\mu_1 D_2v(x_1, x_2), & \text{if } 0 \leq D_1v(x_1, x_2).
\end{cases}
\]

\[
= D_2c(x_1, x_2) + \lambda D_2v(x_1 - 1, x_2 - 1) + \min[0, \mu_2 D_2v(x_1, x_2 + 1)]
\]

\[
+ \begin{cases} 
\mu_1 D_2v(x_1 + 1, x_2), & \text{if } 0 > D_1v(x_1, x_2), \\
\min[0, \mu_1 D_1v(x_1, x_2 + 1)] + \mu_1 D_2v(x_1, x_2), & \text{if } 0 \leq D_1v(x_1, x_2).
\end{cases}
\]

In which the last equality comes from submodular property of \( V \) that \( D_1v(x_1, x_2) \) is a decreasing function of \( x_2 \), so \( D_1v(x_1, x_2 + 1) \leq D_1v(x_1, x_2) < 0 \). Thus all the terms in the last formula are non-positive, which gives \( D_2Tv(x_1, x_2) < 0 \).

Again by Proposition 13 of Chapter 5 (page 218) in Bertsekas (1987) we conclude that the optimal cost function \( J(x_1, x_2) \) is also in \( V_1 \). It implies that \( B_2(x_1) > x_1 \) for \( x_1 < 0 \). Since \( B_2(x_1) \) is an increasing function of \( x_1 \), we have \( B_2(x_1) > \min(0, x_1) \). The same conclusion on \( B_1(x_2) \) is proved similarly.

**Proof of Theorem 3** Let \( V_2 \) be the subset of \( V \) that contains all the functions with the following properties:

\[
D_1J(x_1, x_2) \rightarrow \frac{c_i^+}{\gamma} \text{ uniformly in } x_j \text{ as } x_i \rightarrow \infty, \text{ for } i, j = 1, 2, i \neq j; \quad (6)
\]

\[
D_JJ(x_1, x_2) \rightarrow g_i(x_j | v) \text{ uniformly in } x_j \text{ as } x_i \rightarrow \infty, \text{ for } i, j = 1, 2, i \neq j. \quad (7)
\]

Next, we show that \( V_2 \) is closed under operator \( T \). To simplify the expressions below we denote vectors \((x_1, x_2), (1, 0), \text{ and } (0, 1) \) as \( x, e_1, \text{ and } e_2 \) respectively. Thus \( D_1v(x) = v(x + e_1) - v(x) \), for \( i = 1, 2 \). Given a function \( v \in V_2 \), let\( s \) first consider \( D_1Tv(x) \):

\[
D_1Tv(x) = Tv(x + e_1) - Tv(x)
\]

\[
= D_1c(x) + \lambda D_1v(x - \sum_j e_j) + \sum_j \mu_j D_1v(x) + \mu_1[(R_1v(x + e_1) - R_1v(x))]
\]

\[
+ \mu_2[(R_2v(x + e_1) - R_2v(x))],
\]

where \( R_jv(x) = \min[0, D_jv(x)] \). When \( x_1 \rightarrow \infty \), by (6), the first five terms converges uniformly; by (7), the last two terms converges uniformly and to the same value \( \min[0, g_1(x_2 | v)] \).

Thus, we have the following uniform convergence in \( x_2 \), as \( x_1 \rightarrow \infty \),

\[
D_1Tv(x) \rightarrow \frac{c_1^+}{\gamma} + \frac{\lambda c_1^+}{\gamma} + \frac{\sum_j \mu_j c_1^+}{\gamma}
\]

\[
= \frac{c_1^+}{\gamma}(\gamma + \lambda + \sum_j \mu_j)
\]

\[
= \frac{c_1^+}{\gamma}.
\]
Similarly, we have the following uniform convergence in $x_1$, as $x_2 \to \infty$,

$$D_1Tv(x) \to g_2(x_1|v) + \lambda g_2(x_1 - 1|v) + \sum_j \mu_j g_2(x_1|v)$$

$$+ \mu_1[\min[0, g_2(x_1 + 1|v)] - \min[0, g_2(x_1|v)]]$$

$$:= g_2(x_1|Tv).$$

The other cases of (6) and (7) can be verified in the same way.

In summary we conclude that operator $T$ preserves the properties (6) and (7). Therefore the optimal cost function $J(x_1, x_2)$ is also in $\mathcal{V}_2$. \hfill \Box

Proof of Corollary 1  Theorem 3 states that $D_1J(x_1, x_2)$ converges to $\frac{c^1}{T}$ uniformly in $x_2$. Therefore, there exists an $X_1$ (independent of $x_2$) such that when $x_1 > X_1$, $D_1J(x_1, x_2) > 0$ for all $x_2$, which implies $B_1(x_2) \leq X_1$. The same goes for $B_2(x_1)$. \hfill \Box

Proof of Theorem 4  Let $\mathcal{V}_3$ be the subset of $\mathcal{V}$ that contains all the functions satisfying $D_1J(x_2, x_1) = D_2J(x_1, x_2)$. We want to show that $\mathcal{V}_3$ is closed under operator $T$, which is verified below

$$D_2Tv(x_1, x_2) = Tv(x_1, x_2 + 1) - Tv(x_1, x_2)$$

$$= D_2c(x_1, x_2) + \lambda D_2v(x_1 - 1, x_2 - 1) + (\mu_1 + \mu_2)D_2v(x_1, x_2)$$

$$+ \min[0, \mu_1 D_1v(x_1, x_2 + 1)] - \min[0, \mu_2 D_1v(x_1, x_2)]$$

$$= D_1c(x_2, x_1) + \lambda D_1v(x_2 - 1, x_1 - 1) + (\mu_1 + \mu_2)D_1v(x_2, x_1)$$

$$+ \min[0, \mu_1 D_2v(x_2 + 1, x_1)] - \min[0, \mu_2 D_2v(x_2, x_1)]$$

$$= D_1Tv(x_2, x_1),$$

for any function $v \in \mathcal{V}_3$. Therefore, we can conclude that the optimal cost function $J(x_1, x_2)$ is also in $\mathcal{V}_3$. Then, by the definitions of $B_1(x_2)$ and $B_2(x_1)$, we obtain

$$B_1(x_2) = \min\{x_1 : D_1J(x_1, x_2) \geq 0\}$$

$$= \min\{x_2 : D_2J(x_2, x_1) \geq 0\}$$

$$= B_2(x_2).$$

\hfill \Box